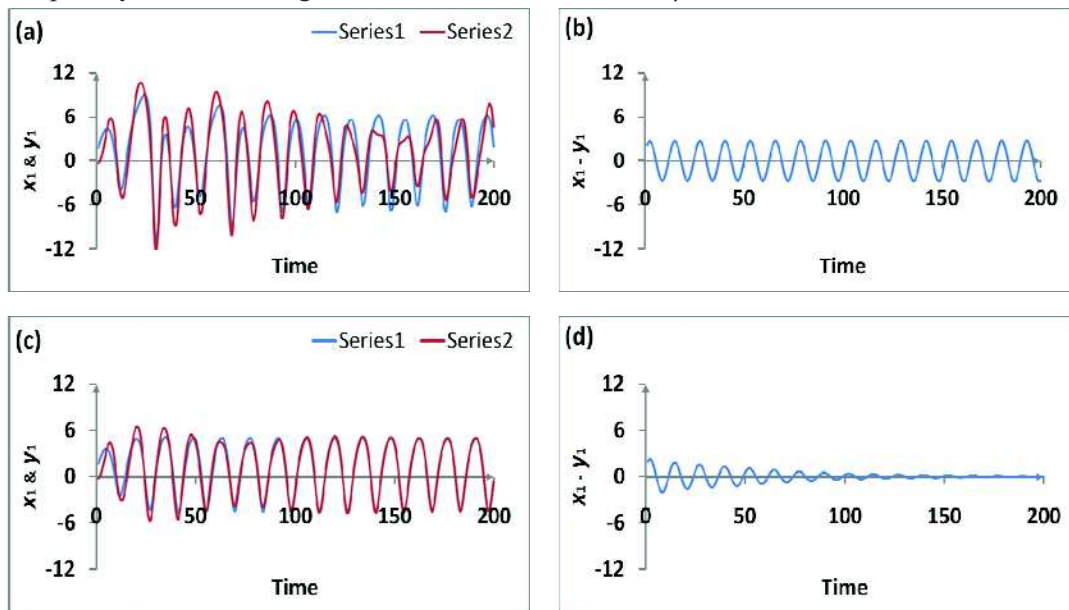


synchronization. Figure 3 shows solutions for  $x_1$ ,  $y_1$  and their differences for various values of  $\alpha$  and  $\beta$ . As illustrated in Fig. 3(d) the difference between derive and response for the values  $\alpha = \beta = 0.96$  is converging to zero in an oscillatory fashion. From numerical results we note that this coupled system is no longer chaotic for values of  $\alpha$  and  $\beta$  less than 0.96.



**Fig. 3** Phase synchronization for coupled chaotic Sprott-L presented by FDEs. (a) shows the solution  $x_1$  as derive (Series1) and  $y_1$  as response (Series2) for  $\alpha = \beta = 1$  and (b) shows the oscillatory difference between derive and response. (c) shows the solution  $x_1$  as derive (Series1) and  $y_1$  as response (Series2) for  $\alpha = \beta = 0.96$  and (d) shows the difference between derive and response that is converging to zero in an oscillatory fashion.

## 4 Conclusion

In this short article we tried to give some examples of nonlinear dynamical systems, chaos and synchronization presented in the form of FDEs. As we have seen, these systems are so sensitive to the fractional order of derivative. Nevertheless, because of non local properties of FDEs the application growth of these systems is inevitable. In particular, in some non smooth domain on which the classical form of ODE is not applicable using FDE has shown remarkable results.

## References

1. G. W. Hill, "On the part of motion of the Lunar Perigee which is a function of the mean motions of the sun and moon", *Acta Math.*, **8** (1886) 1-36.
2. M. Feigenbaum, "Universal behavior in non-linear systems, Los Alamos Sci. (summer) **4** (1980) 27.
3. E.N. Lorenz, "Deterministic non-periodic flow", *J. Atmos. Sci.* **20** (1963) 130.
4. S. Grossmann, "Multicritical phenomena", eds. R. Pynn. and A. T. Skjeltorp, Plenum, New York, 1984.
5. D. Matignon, "Stability results of fractional differential equations with applications to control processing". In: IMACS, IEEE-SMC, Lille, France, (1996) 963.
6. I. Podlubny, *Fractional differential equations*. New York: Academic Press, 1999.
7. K.B. Oldham, Spanier J. The. *New fractional calculus* York: Academic Press, 1974.
8. G.H. Erjaee, Analytical justification of phase synchronization in chaotic systems", *Chaos, Solitons & Fractals* **39** (2009) 1195–1202.
9. B.G. Konopelchenko and V.G. Dubrovsky, "Some new integrable nonlinear evolution equations in (2 + 1) dimensions," *Phys. Lett. A*, **102** (1984) 15–17.
10. M.A. Abdou, "Generalized solitary and periodic solutions for nonlinear partial differential equations by the exp-function method," *Nonlinear Dyn.*, doi:10.1007/s11071-007-9250-1.
11. Tian-lan He, "Bifurcation of traveling wave solutions of (2 + 1) dimensional Konopelchenko Dubrovsky equations," *Applied Mathematics and Computation*, **204** (2008) 773-783.
12. I. Podlubny, "Fractional differential equations", New York, Academic Press, 1999.
13. R.E. Mickens, "Nonstandard finite difference models of differential equations", Singapore, World Scientific, 1994.
14. J.W. Shuaiet, K.W. Wong and L.M. Cheng, *Phys. Rev.* **E56** (1997) 2272.
15. G.H. Erjaee, M. Atabakzadeh and L. M. Saha, " Interesting synchronization like behavior" *Int. J. of Bifurcation and Chaos*, **4** (2004) 1447.
16. L.M. Pecora, and T.L. Carroll, " Synchronization in chaotic systems", *Phys. Rev. Lett.* **64** (1994) 821-824.
17. J. C. Sprott, "Some simple chaotic flows", *Phys. Rev.* **E50**, (1994) R647.

$$\begin{cases} x_1(t_n) = h^\alpha [-x_1(t_{n-1}) + ax_2(t_{n-1})] - \sum_{k=1}^N \left(1 - \frac{1+\alpha}{k}\right) x_1(t_{n-k}) \\ x_2(t_n) = h^\alpha [x_1(t_{n-1}) + x_3^2(t_{n-1})] - \sum_{k=1}^N \left(1 - \frac{1+\alpha}{k}\right) x_2(t_{n-k}) \\ x_3(t_n) = h^\alpha [1 + x_1(t_{n-1})] - \sum_{k=1}^N \left(1 - \frac{1+\alpha}{k}\right) x_3(t_{n-k}), \end{cases} \quad (7)$$

$$\begin{cases} y_1(t_n) = h^\beta [-y_1(t_{n-1}) + ay_2(t_{n-1})] - \sum_{k=1}^N \left(1 - \frac{1+\beta}{k}\right) y_1(t_{n-k}) \\ y_2(t_n) = h^\beta [y_1(t_{n-1}) + y_3^2(t_{n-1})] - \sum_{k=1}^N \left(1 - \frac{1+\beta}{k}\right) y_2(t_{n-k}) \\ y_3(t_n) = h^\beta [1 + x_1(t_{n-1})] - \sum_{k=1}^N \left(1 - \frac{1+\beta}{k}\right) y_3(t_{n-k}). \end{cases} \quad (8)$$

Numerical chaotic results for  $a = -4$  in the  $(x_3, t)$  and  $(y_3, t)$  planes illustrated in Fig. 2. Figure 2(a) shows the phase synchronization for  $\alpha = \beta = 1$ , which is complete agreement with the direct Euler solutions of the original system for Sprott-S ODEs with  $h = 5 \times 10^{-4}$ , and Fig. 2(b) shows the phase synchronization for  $\alpha = \beta = 0.96$  with the same value of  $h$ . As we can see in both figures, the trajectories of the drive and response show that the response attractor is a copy of the drive displaced by some distance in the  $y$  direction. This distance depends on the initial conditions. It is easy to see that the evolution matrix,  $\mathbf{A}$ , in above Sprott-S systems of FDEs takes the form  $\begin{pmatrix} -1 & -4 & 0 \\ 1 & 0 & 2e_3 \\ 0 & 0 & 0 \end{pmatrix}$  which has obviously a zero and two complex eigenvalues  $-1/2 \pm i\sqrt{15}/2$  around  $\mathbf{e} = \mathbf{0}$ .

So we should expect the phase synchronization only between  $x_3$  and  $y_3$ .

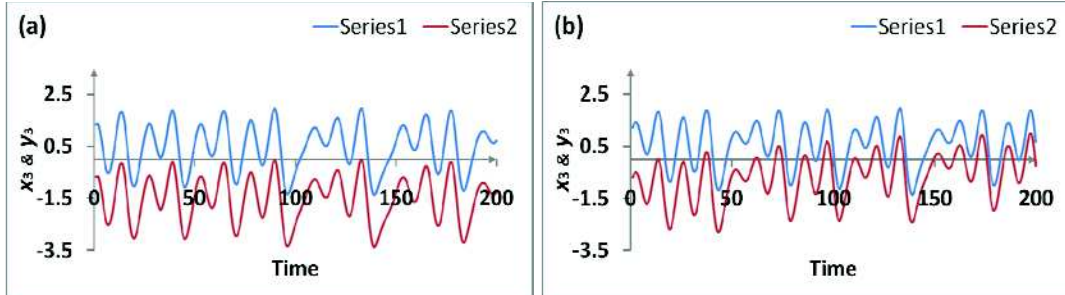
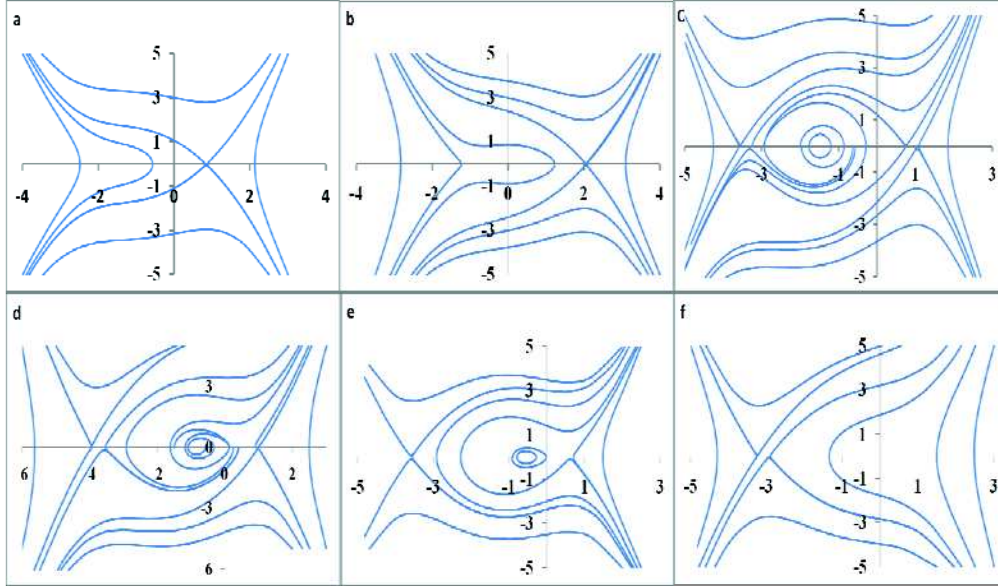


Fig. 2 Phase Synchronization for Coupled chaotic Sprott-S systems presented by FDEs. (a) shows the constant difference between the derive (Series1) and response (Series2) systems for  $\alpha = \beta = 1$  and (b) for  $\alpha = \beta = 0.96$ .

**Example 3** Next consider two Sprott-L systems linked through the second Pecora-Carroll method.

$$\begin{cases} D^\alpha x_1 = x_2 + ax_3 \\ D^\alpha x_2 = bx_1^2 - x_2 \\ D^\alpha x_3 = 1 - x_1, \end{cases} \quad \begin{cases} D^\beta y_2 = x_2 + ay_3 \\ D^\beta y_3 = 1 - y_2. \end{cases} \quad (9)$$

The Sprott-L system presented by ODEs with the parameters  $a = 3.9$  and  $b = 0.9$  is chaotic, and in its above coupled form, the related evolution matrix  $\mathbf{A}$  has two imaginary eigenvalues  $\lambda_{1,2} = \pm\sqrt{3.9}$ . In this case, as illustrated in Fig. 3, phase synchronization between the drive and response occurs in such way that the differences between them will change in oscillatory fashion for different values of  $\alpha$  and  $\beta$ . The frequency of this oscillation depends on the imaginary part of the eigenvalues, but its amplitude is constant depending on the initial values. This phenomenon is called *marginal oscillatory*



**Fig. 2.** Phase portraits of KD equation presented by FDE for derivative order  $\alpha = 0.9$  and different values of parameters  $d$  and  $f$  in different areas shown in the text.

We have tried to take the smallest possible values for the order of derivative  $\alpha$  such that the qualitative behaviors of phase portraits remain unchanged compared to the original phase portraits in the case of  $\alpha = 1$ . We have observed that the different fixed points of the system that exist for the bifurcation values  $d$  and  $f$ , will brake done and will be extremely sensitive for the values of  $\alpha < 0.8$ .

### 3 Applications of FDEs into the Chaos and Synchronizations

Phase synchronization, as special case of synchronization, has been reported for various coupled chaotic systems [14, 15]. This phenomenon occurs when the linearized system describing the evolution of the difference between a pair of chaotic systems has some zero or positive conditional Lyapunov exponents. As we have shown in [15], this behavior also depends upon the eigenvalues of the linearized difference system. More precisely, suppose that two identical chaotic systems  $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}(t))$  and  $\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y}(t))$  are coupled, as drive and response systems, according to the method of Pecora and Carroll [16] by a continuous coupling function  $\mathbf{h}(\mathbf{x})$ . If the system  $\dot{\mathbf{e}} = \mathbf{F}(\mathbf{x}, \mathbf{h}) - \mathbf{F}(\mathbf{y}, \mathbf{h}) = \mathbf{F}(\mathbf{x}, \mathbf{h}) - \mathbf{F}(\mathbf{x} + \mathbf{e}, \mathbf{h})$ , which described the evolution of the difference between two identical systems, has a zero or constant solutions, then the two systems have complete synchronization or phase synchronization, recursively [15]. Indeed, an analysis of the linearized difference system,  $\dot{\mathbf{e}} = \mathbf{A}\mathbf{e}$ , may yield considerable information about the dynamics of the coupled chaotic systems. For the synchronization, we need to determine the conditional Lyapunov exponents of this system, and for the phase synchronization we need to also find the eigenvalues of the system [15]. As shown below, similar results apply to the phase synchronization Sprott systems [17], presented by FDEs. That is, the real parts of the eigenvalues of the evolution matrix,  $\mathbf{A}$ , provide information about the ability to synchronize coupled chaotic systems presented by FDEs. In illustrated numerical results we can see several cases that may arise between derive and response systems in the form of FDEs.

**Example 1** Consider the coupled Sprott-S systems presented by the FDEs

$$\begin{cases} D^\alpha x_1 = -x_1 + ax_2 \\ D^\alpha x_2 = x_1 + x_3^2 \\ D^\alpha x_3 = 1 + x_1 \end{cases} \quad \begin{cases} D^\beta y_1 = -y_1 + ay_2 \\ D^\beta y_2 = y_1 + y_3^2 \\ D^\beta y_3 = 1 + x_1. \end{cases} \quad (6)$$

Using definitions above for FDE, the coupled Sprott-S systems discretized as follows.

$$\begin{cases} -cI^1 D_\xi^\alpha \phi - \phi_{\xi\xi} - 3b\phi^2 + \frac{1}{2}a^2\phi^3 - 3k^2\psi + \frac{3}{2}ak\phi^2 \\ + 3ag_1\phi - g_2 = 0, \\ k\phi + g_1 = \psi. \end{cases} \quad (3)$$

Here,  $g_1$  and  $g_2$  are integral constants. Now we let  $d = \frac{3ak - 6b}{a^2}$ ,  $f = -\frac{2g_2}{a^2}$ ,  $g = -\frac{2c}{a^2}$ ,  $e = \frac{6ag_1 - ck^2}{a^2} - g$ ,  $\phi_\xi = Y$  and  $I^1 D_\xi^\alpha \phi = Z$ . Then (3) reads

$$\begin{cases} Y_\xi = \frac{1}{2}a^2 \left[ \phi^3 + d\phi^2 + e\phi + f - \frac{2c}{a^2}Z \right] \\ Z_\xi = D_\xi^\alpha \phi \\ \phi_\xi = Y. \end{cases} \quad (4)$$

In this article we will use the same sets of bifurcation parameters as those used in [11] for system (4) in order to investigate the qualitative behavior of the phase portraits for system (3) with varying values of derivative order  $\alpha$ . Here, we suppose  $a = 1$  and fixed  $e$  as a positive number, say 1, and we choose  $d$  and  $f$  as bifurcation parameters. As discussed in [11], the qualitative behavior of phase portraits for zero or negative values of  $e$  is similar to the ones with positive  $e$ . So we can vary only  $d$  and  $f$ . Here we shall use the Grunwald-Letnikov method [12] to discretize system (4). In this

method fractional derivative is discretized as  $D^\alpha \mathbf{x}(t) = \sum_{k=0}^{\lfloor t_n/h \rfloor} c_k^\alpha \mathbf{x}(t_{n-k})$ , where  $h$  is the step size,

$\lfloor t_n/h \rfloor$  denotes the integer part of  $t_n/h$ ,  $t_n = nh$  and  $c_k^\alpha$  are the Grunwald-Letnikov coefficients

defined by  $c_k^\alpha = h^{-\alpha} (-1)^k \binom{\alpha}{k}$ ,  $k = 0, 1, 2, \dots$ . These coefficients can also be evaluated recursively

by  $c_0^\alpha = h^{-\alpha}$  and  $c_k^\alpha = \left(1 - \frac{1+\alpha}{k}\right) c_{k-1}^\alpha$ ,  $k = 1, 2, 3, \dots$ . Using this method system (4) discretized as

follow.

$$\begin{cases} Z(t_n) = \sum_{k=1}^N \left(1 - \frac{1+\alpha}{k}\right) \phi(t_{n-k}) \\ \phi(t_n) = \phi(t_{n-1}) + h y(t_{n-1}) \\ y(t_n) = y(t_{n-1}) + 0.5h \left( \phi(t_{n-1})\phi^2(t_n) + d\phi(t_{n-1})\phi(t_n) \right. \\ \left. + \phi(t_n) + f - 2c Z(t_n) \right). \end{cases} \quad (5)$$

Note that in this discretization we have used the non-standard Mickens' method [13] to obtain a stronger result. This means that in the discretization process we have replaced  $\phi^3(t_n)$  with  $\phi^2(t_{n-1})\phi(t_n)$  or  $\phi^2(t_n)$  with  $\phi(t_{n-1})\phi(t_n)$ . To be consistence with [11] we have chosen the same bifurcation parameters  $d$  and  $f$  from the different regions. Now, after solving the discretized system (5) the results are illustrated in Fig. 1. Each figure in the set of Fig. 1 represents the phase portrait for different regions of parameters. In these figures we have taken the order of derivative  $\alpha = 0.9$ . As we can see in these figures the qualitative behavior of the saddle points did not show significant change, comparing with their counterpart figures in [11]. This is the same for the cusp points. On the other hand the center points are very sensitive to the value of derivative order  $\alpha$ .

chaos and fractals. Such this discovery is having a major impact on almost all fields of research such as: science, engineering, mathematics, even in physiology and people issues. The discovery of chaos changes our understanding of the foundations of physics, and has many practical applications as well. Hence, chaotic motion is not a rare phenomenon. Indeed, in any dynamical system which can be described by a set of first order differential equations, several necessary conditions for chaotic motion are needed. First, the system should have at least three independent dynamical variables, and second the equations of motion should contain a nonlinear term, that couples several of the variables. Despite the fact that, many of dynamical systems which described a real physical phenomenon and satisfy in above conditions may show chaotic motion, these conditions do not guarantee chaos, they do make its existence possible.

Evidently, we are not able to discuss the history and huge applications of chaos in this short article. Instead, we will state some examples of dynamical systems and chaos which are not presented in the classical form of nonlinear systems of ODE, but in the form of fractional differential equations (FDEs). Indeed, study of the FDEs as a dynamical system is a novel and appealing subject which has motivated the leading research literatures in recent years. For example see [5-8] and references cited therein. The non-local nature property of the FDEs is a distinguish property. In fact, a local operator, such as an integer order differential equation, has the property that only the present state of a system can determine its coming state, so this operator is oblivious to the history of the system. On the other hand, the so-called non-local property means that the next state of a system not only depends upon its current state but also upon its historical states starting from the initial time. This, of course, is closer to reality and is therefore the main reason that FDEs have become more and more popular and have been applied to dynamical systems.

The synchronization phenomenon is an interesting and well known property of the chaotic systems. In this article the existence of phase synchronization, which is slightly different from synchronization, is investigated in deferent coupled chaotic fractional differential equations.

## 2 Applications of FDEs into the Dynamical Systems

As an example in application of FDEs to the dynamical systems, we choose Konopelchenko–Dubrovsky (KD) equation. They introduced their (2+1)-dimensional model equation in 1984 [9]. Extensive efforts have been devoted to solving and analyzing the qualitative behavior of the solutions and to discussing the bifurcation behavior of this equation for varying parameters. While no unified method to attack this nonlinear problem has appeared, many appealing methods have been utilized to determine exact solutions of the KD equation. The Exp-function method [10] and the inverse scattering transform method [9] are among these. We consider the KD equation presented by FDE in time derivative

$$\begin{cases} D_t^\alpha u - u_{xxx} - 6buu_x + \frac{3}{2}a^2u^2u_x - 3v_y + 3au_xv = 0, \\ u_y = v_x, \end{cases} \quad (1)$$

where  $a, b$  are real parameters,  $D_t^\alpha \mathbf{x}(t) = J^{n-\alpha} D_t^n \mathbf{x}(t)$  is the  $n^{\text{th}}$  order Riemann-Liouville integral

operator defined by  $J^n \mathbf{x}(t) = \frac{1}{\Gamma(n)} \int_0^t (t-\tau)^{n-1} \mathbf{x}(\tau) d\tau$ , with  $0 < \alpha \leq 1$  and  $D_t^n(\cdot)$  being ordinary

derivative of order  $n$  for time  $t > 0$ . Following the procedure in [11] for ODE case, we let  $\xi = x + ky - ct$ , with  $c$  as a wave speed and  $k$  as a parameter. It follows that  $u(x, y, t) = \phi(\xi)$  and  $v(x, y, t) = \psi(\zeta)$ . Substituting this transformation into system (1) yields,

$$\begin{cases} -cD_\xi^\alpha \phi - \phi_{\xi\xi\xi} - 6b\phi\phi_\xi + \frac{3}{2}a^2\phi^2\phi_\xi - 3k\psi + \frac{3}{2}ak\phi^2 = 0 \\ k\phi_\xi = \psi_\xi. \end{cases} \quad (2)$$

Integrating both sides of system (2) with respect to  $\xi$  leads to

# Application of Fractional Calculus in Chaos and Synchronization

G. H. Erjaee<sup>1</sup>

Mathematics Department  
Shiraz University, Shiraz, Iran

**Abstract.** It is well known that the extreme sensitivity to the initial conditions mathematically present in the non-linear systems, studied by Poincaré in 1920, has come to be called dynamical instability, or simply *chaos*. So, the chaotic system is sensitive to initial conditions, and hence, is unpredictable over a large time scale. On the other hand, recently fractional calculus is increasingly applied to the dynamical systems and chaos. In this article, we will discuss some interesting applications of fractional calculus in dynamical systems. Synchronization in coupled chaotic systems is another interesting phenomenon. We will also discuss the synchronization in some coupled chaotic systems presented by fractional differential equations.

*Keywords:* Chaos, synchronization and fractional differential equations.

## 1 Introduction

In our everyday life, we experience a number of natural changes around us. All such changes (or evolutions), are governed by some principles (rules) of nature. If such rules can be written in the form of mathematical equations, a large number of behaviors can be explained or predicted in advance. However, until this stage of modern science, we have succeeded very little in modeling, mathematically, all such natural phenomena. This is because almost all natural phenomena are nonlinear in nature and behave with their own independent laws. No definite common principle has been discovered until now which explains a nonlinear phenomena. Complexity may emerge in every nonlinear system. Actually complexity is the measure of the property *nonlinearity* of system.

Mathematical dynamical systems theory, as a tool to study complexity of the systems, had its inception with Newton. The search seems to have begun systematically only after the invention of differential equations, and discovery of the Laws of motion by Newton. However, this search was not totally vain as it led to the development of a totally new approach to the problems. It was Poincaré who emphasized the need for qualitative approach to the problems rather than the quantitative ones. So Poincaré was the first person to acknowledge the possibility of complexity and irregularity. Nonlinear aspects of classical physics were first viewed, in the 19-century, as a set of phenomena, which occur at the limit of validity of linear laws & linear models. By the end of the 19<sup>th</sup> century, however, accurate observations in different non-linear systems had been accumulated over a long enough period to enable one to recognize as well as to measure the effects of the basic complexity and nonlinearity of Newton's Law of attraction. Poincaré (1921, 1957), Lindstedt (1882), Hill (1886) [1], and Lyapunov (1907) made significant contributions to the complexity and nonlinearity theory.

Chaos is one of the most important properties of some complex systems. Indeed, study of deterministic chaos by iterated maps goes back to Henry Poincaré, but did not become a part of theoretical physics until after M. Feigenbaum's (1978, 1980) [2] discovery and analysis by a renormalization group method of universal critical exponents at the transition to chaos in a one dimension maps. Lorenz (1963) [3] and Grossmann (1984) [4] have also studied these maps as paradigms of chaos in higher dimensional systems. Since the discovery of universality of transition to chaos, and the observation of period doubling sequences much has been written about deterministic

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<sup>1</sup> The author is also a research fellow of Qatar University, Doha, Qatar  
E-mail: erjaee@shirazu.ac.ir