AN EFFICIENT APPROACH FOR SOLVING A WIDE CLASS OF FUZZY LINEAR PROGRAMMING PROBLEMS

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Abstract. In this paper we are going to introduce an efficient approach for solving a wide class of fuzzy linear programming (FLP) problems. Zimmermann first used the max-min operator of Bellman and Zadeh to solve FLP problems. Many researchers use his approach to solving different types of fuzzy programming. We first state some comments concerning Zimmermann approach and show some inefficiencies of it. Then we suggest some modifications for his approach. Finally, after defining a smooth membership function, we present a new approach by using max-product operator that is a very effective. Numerical examples demonstrate that the proposed approach can be effectively incorporated with other approaches to FLP.

keywords : Fuzzy mathematical programming, Aspiration levels, max-min operator, max-product operator.

1. Introduction

Fuzzy linear programming (FLP) is a very useful and practical model for many real world problems. Concept of decision analysis in fuzzy environment was first proposed by Bellman and Zadeh [1]. Zimmermann [2, 3] first used the max-min operator of Bellman and Zadeh to solve FLP problems. Other researchers used this operator, too. See [4, 5]. Max-min operator has been used in solving other types of fuzzy programming. For example See [6]-[11]. But there are some difficulties when they used this operator in their papers or their books. On the other hand, min operator has some weakness when it used as a t-norm. In this paper we investigate difficulties and weakness of max-min operator and suggest a more deficiency approach for solving a wide class of the fuzzy programming problems. Finally we have solved some numerical examples and we have shown the efficiency of our approach.

2. Some comments concerning Zimmermann approach

Consider the following symmetric fuzzy linear programming problem

$$\begin{aligned} c^T\mathbf{x} &\,\widetilde{\geq}\, z_0 \\ A\mathbf{x} &\,\widetilde{\leq}\, b \\ \mathbf{x} &\geq 0 \end{aligned} \tag{2.1}$$

Where $\widetilde{\leq}(\widetilde{\geq})$ denote the fuzzified version of $\leq$ $(\geq)$. Many FLP problems may be transformed into (2.1). We may write (2.1) in the following form:

$$\begin{pmatrix} -c^T \\ A \end{pmatrix} \mathbf{x} \,\widetilde{\leq}\, \begin{pmatrix} -z_0 \\ b \end{pmatrix}$$

By assumption

$$B = \begin{pmatrix} -c^T \\ A \end{pmatrix} \quad , \quad d = \begin{pmatrix} -z_0 \\ b \end{pmatrix}$$

(2.1) may be written as follows:

$$\begin{aligned} B\mathbf{x} &\tilde{\leq} d \\ \mathbf{x} &\geq 0 \end{aligned} \tag{2.2}$$

Each of the $(m+1)$ rows of model (2.2) could be represented as a fuzzy set, with membership functions $\mu_i(\mathbf{x}), i = 1, 2, \ldots, m+1$. By using the fuzzy decision proposed by bellman and Zadeh [1], we have

$$\mu_{\tilde{D}}(X) = \min\{\mu_i(\mathbf{x}) \ , i = 1, 2, \ldots, m+1\} \tag{2.3}$$

$\mu_i(\mathbf{x})$ can be interpreted as the degree to which $\mathbf{x}$ fulfills (satisfies) the fuzzy inequality $B_i\mathbf{x} \leq d_i$ (Where B_i denotes the ith row of B).

Assume that the decision maker is not interested in a fuzzy set but in a crisp "optimal" solution, then we could suggest the "maximizing solution" to equation (2.3), which is the solution to the possibly nonlinear programming problem

$$\max_{\mathbf{x}\geq 0} \ \mu_{\tilde{D}}(\mathbf{x}) = \max_{\mathbf{x}\geq 0} \ \min_i\{\mu_i(\mathbf{x}) \ , i = 1, 2, \ldots, m+1\} \tag{2.4}$$

The simplest type of membership function (piecewise linear function) is as follows:

$$\mu_i(\mathbf{x}) = \begin{cases} 1 & B_i\mathbf{x} \leq d_i \\ 1 - \frac{B_i\mathbf{x} - d_i}{p_i} & d_i < B_i\mathbf{x} \leq d_i + p_i \\ 0 & B_i\mathbf{x} > d_i + p_i \end{cases} \tag{2.5}$$

Where p_i is the tolerance of ith constraint $i = 1, 2, \ldots, m+1$.
With Zimmermann's approach [3], using max-min operator, a max-min model for problem (2.4), can be stated as follows:

$$\begin{aligned} &\max \ \lambda \\ &\ s.t. \\ &\qquad 1 - \frac{B_i\mathbf{x} - d_i}{p_i} \geq \lambda \ , \ \ i = 1, 2, \ldots, m+1 \\ &\qquad \mathbf{x} \geq 0 \end{aligned} \tag{2.6}$$

Therefore, we have

$$\begin{aligned} &\max \ \lambda \\ &\ s.t. \\ &\qquad \lambda p_i + B_i\mathbf{x} \leq d_i + p_i \ , \ \ i = 1, 2, \ldots, m+1 \\ &\qquad \mathbf{x} \geq 0 \end{aligned} \tag{2.7}$$

Now consider example 13-5 in [3].

Example 2.1.

$$
\begin{aligned}
\min\ & 41400x_1 + 44300x_2 + 48100x_3 + 49100x_4 \\
s.t.\ & \\
& 0.48x_1 + 1.44x_2 + 2.16x_3 + 2.4x_4 \geq 170 \\
& 16x_1 + 16x_2 + 16x_3 + 16x_4 \geq 1300 \\
& x_1 \geq 6 \\
& x_2, x_3, x_4 \geq 0
\end{aligned} \tag{2.8}
$$

The solution is $x_1 = 6, x_2 = 16.29, x_3 = 0, x_4 = 58.96$ *and Min cost=3864975.*

Then he added some lower bounds and spread of the tolerance interval as follows:

$$
\begin{array}{llll}
d_1 = 3700000 & d_2 = 170 & d_3 = 1300 & d_4 = 6 \\
p_1 = 500000 & p_2 = 10 & p_3 = 100 & p_4 = 6
\end{array}
$$

After dividing all rows by their respective p_i's and rearranging in such a way that only λ remain on the left-hand side, we have:

$$
\begin{aligned}
\max\ & \lambda \\
s.t.\ & \\
& 0.083x_1 + 0.089x_2 + 0.096x_3 + 0.098x_4 + \lambda \leq 8.4 \\
& 0.048x_1 + 0.144x_2 + 0.216x_3 + 0.24x_4 - \lambda \geq 17 \\
& 0.16x_1 + 0.16x_2 + 0.16x_3 + 0.16x_4 - \lambda \geq 13 \\
& 0.167x_1 - \lambda \geq 1 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned} \tag{2.9}
$$

Obtained optimal solution in [3] is:

$$x_1 = 17.414 \quad , \quad x_2 = 0 \quad , \quad x_4 = 66.54 \quad , \quad z^* = 3988250$$

But, by transform (2.8) into (2.2), the original problem is as follows:

$$
\begin{aligned}
& 41400x_1 + 44300x_2 + 48100x_3 + 49100x_4 \widetilde{\leq} 3700000 \\
& -0.48x_1 - 1.44x_2 - 2.16x_3 - 2.4x_4 \widetilde{\leq} -170 \\
& -16x_1 - 16x_2 - 16x_3 - 16x_4 \widetilde{\leq} -1300 \\
& -x_1 \widetilde{\leq} -6 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

Therefore, according to (2.5), membership functions are as follows:
For the first tolerance

$$
\mu(z_1) = \begin{cases} 1 & z_1 \leq 37 \\ 1 - \frac{z_1 - 37}{5} & 37 \leq z_1 \leq 42 \\ 0 & z_1 \geq 42 \end{cases}
$$

$$\textit{where} \qquad z_1 = 0.414x_1 + 0.443x_2 + 0.481x_3 + 0.491x_4$$

For the second tolerance

$$\mu(z_2) = \begin{cases} 1 & z_2 \leq -170 \\ 1 - \frac{z_2+170}{10} & -170 \leq z_2 \leq -160 \\ 0 & z_2 \geq -160 \end{cases}$$

$$\textit{where} \qquad z_2 = -0.48x_1 - 1.44x_2 - 2.16x_3 - 2.4x_4$$

For the tertiary tolerance

$$\mu(z_3) = \begin{cases} 1 & z_3 \leq -1300 \\ 1 - \frac{z_3+1300}{100} & -1300 \leq z_3 \leq -1200 \\ 0 & z_3 \geq -1200 \end{cases}$$

$$\textit{where} \qquad z_3 = -16x_1 - 16x_2 - 16x_3 - 16x_4$$

For the fourth tolerance

$$\mu(z_4) = \begin{cases} 1 & z_4 \leq -6 \\ 1 - \frac{z_4+6}{6} & -6 \leq z_4 \leq 0 \\ 0 & z_4 \geq 0 \end{cases}$$

$$\textit{where} \qquad z_4 = -x_1$$

After dividing all rows by their respective p_i's and rearranging in such a way that only λ remain on the left-hand side, we have:

$$(2.10) \qquad \begin{aligned} & \max\ \lambda \\ & s.t. \\ & 0.083x_1 + 0.089x_2 + 0.096x_3 + 0.098x_4 + \lambda \leq 8.4 \\ & 0.048x_1 + 0.144x_2 + 0.216x_3 + 0.24x_4 - \lambda \geq 16 \\ & 0.16x_1 + 0.16x_2 + 0.16x_3 + 0.16x_4 - \lambda \geq 12 \\ & 0.167x_1 - \lambda \geq 0 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

So the correct solution of the problem is

$$x_1 = 4.6794 \quad , \quad x_2 = 15.5447 \quad , \quad x_3 = 0 \quad , \quad x_4 = 59.66$$
$$z^* = 3881663 \quad , \quad \lambda^* = 0.7815$$

Therefore, he mistakes in using his approach.
On the other hand, he did not consider the constraints

$$d_i \leq B_i\mathbf{x} \leq d_i + p_i$$

as defined in (2.5). Note that the optimal value of (2.4) is in [0,1]. But if for example $\mathbf{x}$ be a solution where $B_i\mathbf{x} = \frac{d_i}{2}$ we have

$$1 - \frac{B_i\mathbf{x} - d_i}{p_i} = 1 - \frac{\frac{d_i}{2} - d_i}{2} = 1 + \frac{d_i}{4} > 1$$

Therefore, it may be $\lambda^* > 1$ in (2.7). Indeed, without constraints $d_i \leq B_i\mathbf{x} \leq d_i + p_i$, there is no upper bound for

$$1 - \frac{B_i\mathbf{x} - d_i}{p_i}$$

Although some researchers try to remove this error by adding the constraint $0 \leq \lambda \leq 1$, (see [5, 2]), but it is not an upper bound for membership functions. Thus, we must solve the following problem instead of solving (2.7)(as page 290 in [3])

$$
\begin{aligned}
&\max\ \lambda \\
&s.t. \\
&\quad \lambda p_i + B_i\mathbf{x} \leq d_i + p_i\ ,\quad i = 1, 2, \ldots, m+1 \\
&\quad B_i\mathbf{x} \leq d_i + p_i\quad ,\quad i = 1, 2, \ldots, m+1 \\
&\quad \mathbf{x}, \lambda \geq 0
\end{aligned} \tag{2.11}
$$

Note 2.2. *It is clear that the problem has been solved in fuzzy environment. So the obtained optimal solution of (2.10) has a degree of aspiration (because in fuzzy environment any solution has a degree of aspiration in [0,1]). Therefore, we must indicate the solution x^* with a aspiration level λ^*.*

However, min operator has some weakness when it used as a t-norm. When min operator use as a t-norm, the sets with higher membership values, with respect to other sets, do not have any effect in intersection of fuzzy sets. For more explicit statement, consider two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$, such that $\widetilde{A} = \{(a, 0.3), (b, 0.5)\}$ and $\widetilde{B} = \{(a, 0.4), (b, 0.6)\}$. By min operator as a t-norm $\widetilde{A} \cap \widetilde{B} = \widetilde{A}$. Now, assume that $\widetilde{B} = \{(a, 0.5), (b, 0.7)\}$, on the other word if $\mu_{\widetilde{B}}(a) \geq 0.4$ and $\mu_{\widetilde{B}}(b) \geq 0.6$ then $\widetilde{A} \cap \widetilde{B} = \widetilde{A}$. Indeed, the degrees of the members in $\widetilde{B}$, do not any effect in intersection. Therefore, when max-min operator use on solving a FLP, some constraints may be disregarded. As a example, consider the following membership functions:

$$
\mu_1(\mathbf{x}) = \begin{cases} 1 & 7 \leq \mathbf{x} \\ 0.06(\mathbf{x} - 6) + 0.94 & 6 \leq \mathbf{x} \leq 7 \\ 0.08(\mathbf{x} - 3) + 0.7 & 3 \leq \mathbf{x} \leq 6 \\ 0.35(\mathbf{x} - 1) & 1 \leq \mathbf{x} \leq 3 \\ 0 & \mathbf{x} < 1 \end{cases} \tag{2.12}
$$

$$
\mu_2(\mathbf{x}) = \begin{cases} 1 & 8 \leq \mathbf{x} \\ 0.11(\mathbf{x} - 6) + 0.78 & 6 \leq \mathbf{x} \leq 8 \\ 0.3(\mathbf{x} - 4) + 0.18 & 4 \leq \mathbf{x} \leq 6 \\ 0.06(\mathbf{x} - 1) & 1 \leq \mathbf{x} \leq 4 \\ 0 & \mathbf{x} < 1 \end{cases} \tag{2.13}
$$

$$
\mu_3(\mathbf{x}) = \begin{cases} 1 & 9 \leq \mathbf{x} \\ 0.14(\mathbf{x} - 7) + 0.72 & 7 \leq \mathbf{x} \leq 9 \\ 0.04(\mathbf{x} - 4) + 0.6 & 4 \leq \mathbf{x} \leq 7 \\ 0.6(\mathbf{x} - 3) & 3 \leq \mathbf{x} \leq 4 \\ 0 & \mathbf{x} < 3 \end{cases} \tag{2.14}
$$

These membership functions and their minimum have been shown in figure (1). It is obvious that μ_1 does not have any influence in μ_{min}. We prefer to use max-product operator on solving a FLP instead max-min operator because product operator as a t-norm has the compensation property. But for this purpose we must first make a single variable membership function. the reason of this proceeding would be stated in section 4.

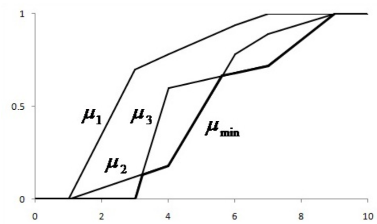

FIGURE 1. membership functions of $\mu_1, \mu_2, \mu_3, \mu_{min}$

3. Making a single formula membership function

Piecewise linear membership functions are often as be shown in figure (2). In this section, we try to introduce a single formula for defining a function which is very close to many membership functions.
Membership functions can be classified in three groups. triangular shape, left or right half-trapezoidal shape and trapezoidal shape. Here, we obtain a approximate function with single formula for all types of mentioned shapes in figure (2). For this purpose, we choose several main points that have been given by decision maker or we deduce from the membership function.
Figure (3) shows a trapezoidal membership function with several main points.
Therefore, we require a suitable membership function to pass from through these main points. We consider this function as follows:

$$\mu^*(\mathbf{x}) = \frac{1}{\pi} \arctan\left(p(\mathbf{x})\right) + \frac{1}{2} \tag{3.1}$$

Where, $p(\mathbf{x})$ is a polynomial that fit some points. It is clear that the rang of this function is $[0, 1]$. Because, the rang of the function arctan *or* tan^{-1} is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. With some simple changes it can be transformed to $[0, 1]$, such as:

$$-\frac{\pi}{2} \leq \arctan(p(\mathbf{x})) \leq \frac{\pi}{2}$$
$$0 \leq \frac{\pi}{2} + \arctan(p(\mathbf{x})) \leq \pi$$
$$0 \leq \frac{1}{2} + \frac{1}{\pi} \arctan(p(\mathbf{x})) \leq 1$$

Now, we introduce $p(\mathbf{x})$. First, Let $\mu(\mathbf{x})$ be a membership function with left or right half-trapezoidal shape.
Consider a polynomial as follows:

$$p(\mathbf{x}) = a_0 + a_1\mathbf{x} + a_3\mathbf{x}^3 + \cdots + a_{2n-1}\mathbf{x}^{2n-1} \tag{3.2}$$
$$\textit{then} \quad p'(\mathbf{x}) = a_1 + 3a_3\mathbf{x}^2 + \cdots + (2n-1)a_{2n-1}\mathbf{x}^{2n-2}$$

Here, if we assume that $a_1, a_3, \ldots, a_{2n-1}$ are non-negative (non-positive) numbers then $p'(\mathbf{x}) \geq 0$ ($p'(\mathbf{x}) \leq 0$) for all ($\mathbf{x} \in \mathbb{R}$), it means that $p(\mathbf{x})$ and thereupon $arctan(p(\mathbf{x}))$ is increasing (decreasing) function on $\mathbb{R}$. Therefore, we may approximate any half-trapezoidal membership function which is increasing (decreasing) by such a polynomial. Note that the left(right) half-trapezoidal functions are increasing (decreasing).
On the other hand, if $\mu(\mathbf{x})$ be a membership function with trapezoidal or triangular

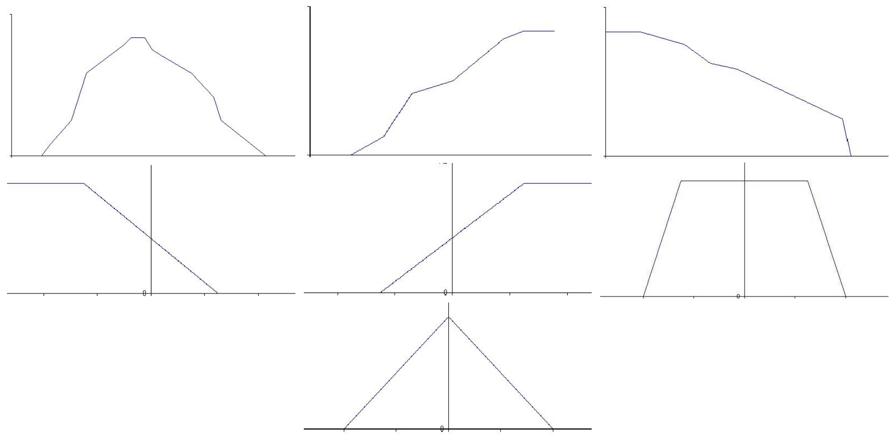

FIGURE 2. Different types of membership functions

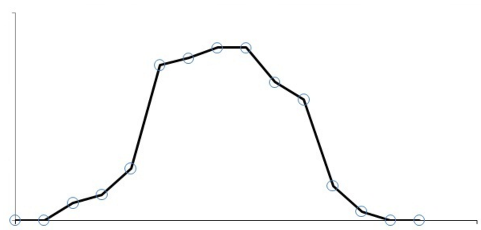

FIGURE 3. A trapezoidal membership function with several main points

shape, consider a polynomial as follows:

$$p(\mathbf{x}) = a_0 + a_2(\mathbf{x}-\mathbf{x}_0)^2 + a_4(\mathbf{x}-\mathbf{x}_0)^4 + \cdots + a_{2n}(\mathbf{x}-\mathbf{x}_0)^{2n} \tag{3.3}$$

$$\textit{then} \qquad p'(\mathbf{x}) = 2a_2(\mathbf{x}-\mathbf{x}_0) + 4a_4(\mathbf{x}-\mathbf{x}_0)^3 + \cdots + 2na_{2n}(\mathbf{x}-\mathbf{x}_0)^{2n-1}$$

Where x_0 is a point that $\mu(\mathbf{x})$ is increasing before x_0 and decreasing after it. If coefficients be non-positive, this polynomial is increasing for $\mathbf{x} < \mathbf{x}_0$ and decreasing for $\mathbf{x} > \mathbf{x}_0$. Because $p'(\mathbf{x})$ is non-negative for $\mathbf{x} < \mathbf{x}_0$ and non-positive for $\mathbf{x} > \mathbf{x}_0$. Also, maximum or minimum of $p(\mathbf{x})$ occurs in $\mathbf{x}_0$. Thus, it is suitable for trapezoidal and triangular membership functions.
The degree of the polynomials is dependent to the number of main points.

Now, we explain this procedure. Assume that S be a set of main points as below:

$$S = \{(\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x}_2,\mathbf{y}_2),\ldots,(\mathbf{x}_m,\mathbf{y}_m)\}$$

Where, $\mathbf{y}_i$ is considered as membership value for $\mathbf{x}_i$ and $0 < \mathbf{y}_i < 1$.
Thus, according to the (3.1), $p(\mathbf{x}_i)$ should be considered as follows:

$$p(\mathbf{x}_i) = \tan\left(\pi \times \left(\mathbf{y}_i - \frac{1}{2}\right)\right) \qquad , \qquad i = 1,2,\ldots,m \tag{3.4}$$

Note that, for every $\mathbf{y}_i$ obtain a unique value for $p(\mathbf{x}_i)$ in $(-\infty, \infty)$. Therefore, the coefficients a_j, will be obtain by solving the system of equations (3.4) Or equivalently by solving a nonlinear programming as follows:

$$\begin{aligned} &\min \sum_{i=1}^{m} |p(\mathbf{x}_i) - \mathbf{b}_i| \\ &s.t. \\ &\quad a_j \geq 0 \quad j = 1, 3, \ldots, 2n-1 \end{aligned} \tag{3.5}$$

Where $\mathbf{b}_i = \tan\left(\pi \times \left(\mathbf{y}_i - \frac{1}{2}\right)\right)$ are deterministic values and $p(\mathbf{x})$ consider as (3.2).

Note 3.1. *When the membership function is decreasing (half-trapezoidal shape), we should assume $a_j \leq 0$ for all $j = 1, 3, \ldots, 2n-1$.*

And

$$\begin{aligned} &\min \sum_{i=1}^{m} |p(\mathbf{x}_i) - \mathbf{b}_i| \\ &s.t. \\ &\quad a_j \leq 0 \quad j = 2, 4, \ldots, 2n \end{aligned} \tag{3.6}$$

Where $p(\mathbf{x})$ consider as (3.3).

Above problem can be transformed to a linear programming problem. Let

$$t_i = p(\mathbf{x}_i) - \mathbf{b}_i \qquad i = 1, 2, \ldots, m$$

Here t_i are free variables. By setting $t_i = t_i^+ - t_i^-$ where $|t_i| = t_i^+ + t_i^-$ and $t_i^+, t_i^- \geq 0$ (3.5) transform to:

$$\begin{aligned} &\min \sum_{i=1}^{m} (t_i^+ + t_i^-) \\ &s.t. \\ &\quad p(\mathbf{x}_i) - t_i^+ + t_i^- = \mathbf{b}_i \\ &\quad a_j \geq 0 \quad j = 1, 3, \ldots, 2n-1 \\ &\quad t_i^+, t_i^- \geq 0 \end{aligned} \tag{3.7}$$

And (3.6) transform to::

$$\begin{aligned} &\min \sum_{i=1}^{m} (t_i^+ + t_i^-) \\ &s.t. \\ &\quad p(\mathbf{x}_i) - t_i^+ + t_i^- = \mathbf{b}_i \\ &\quad a_j \leq 0 \quad j = 2, 4, \ldots, 2n \\ &\quad t_i^+, t_i^- \geq 0 \end{aligned} \tag{3.8}$$

By solving these problems, we obtain coefficients a_j and consequently obtain $p(\mathbf{x})$.

Now, we illustrate this procedure with some examples:

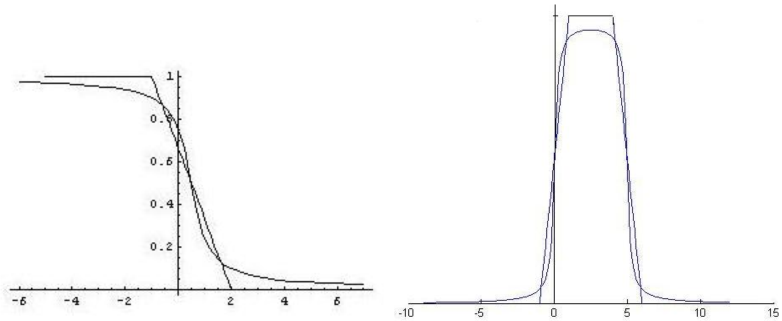

FIGURE 4. smooth membership functions are good approximate for piecewise linear functions

Example 3.2. *Consider the following function:*

$$\mu(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \leq -1 \\ \frac{2-\boldsymbol{x}}{3} & -1 < \boldsymbol{x} \leq 2 \\ 0 & \boldsymbol{x} > 2 \end{cases}$$

It is clear that $\mu(\boldsymbol{x})$ is decreasing. we choose main points as the set

$$S = \{(-1, 3.078), (2, -3.078)\}$$

*(**Note:** after modifying rang of function, -3.078 and 3.078 transform to 0.1 and 0.9, respectively). By solving a linear programming such as (3.6), when $a_1 \leq 0$ we have:*

$$p(\boldsymbol{x}) = -2.052\boldsymbol{x} + 1.026$$

Thus, corresponding smooth membership function, $\mu^*(\boldsymbol{x})$, according to (3.1) is as follows:

$$\mu^*(\boldsymbol{x}) = \frac{1}{\pi}\arctan(-2.052\boldsymbol{x} + 1.026) + \frac{1}{2}$$

Figure (4) shows $\mu(\boldsymbol{x})$ and $\mu^*(\boldsymbol{x})$.

Example 3.3. *Consider trapezoidal membership function:*

$$\mu(\boldsymbol{x}) = \begin{cases} 1 & 1 < \boldsymbol{x} \leq 4 \\ \frac{6-\boldsymbol{x}}{2} & 4 < \boldsymbol{x} \leq 6 \\ \frac{\boldsymbol{x}+1}{2} & -1 < \boldsymbol{x} \leq 1 \\ 0 & \boldsymbol{x} > 6\ ,\ \ \boldsymbol{x} \leq -1 \end{cases}$$

Main points and corresponding membership values stated in the below table:

$\boldsymbol{x}$	-1	1	2.5	4	6
$\mu(\boldsymbol{x})$	0.05	0.9	0.95	0.9	0.05
$p(\boldsymbol{x})$	-6.314	3.078	6.314	3.078	-6.314

By using (3.3) with $n = 3$, after solving a linear programming such as (3.7) we have:

$$p(\boldsymbol{x}) = 6.314 - 1.031(\boldsymbol{x} - 2.5)^2$$

*(**Note:** here, obtained value for a_4 and a_6 is equal to zero). Therefore*

$$\mu^*(\boldsymbol{x}) = \frac{1}{\pi}\arctan(6.314 - 1.031(\boldsymbol{x} - 2.5)^2) + \frac{1}{2}$$

Figure (4) show $\mu(\boldsymbol{x})$ and $\mu^*(\boldsymbol{x})$.

4. Max-Product approach for solving FLP

In this section, we are going to use algebraic product t-norm for FLP problems. But obtained function by using this t-norm for piecewise linear membership functions is very complex. for example consider $\mu_2(\mathbf{x}), \mu_3(\mathbf{x})$ defined in (2.13,2.14). By using product operator as a we have:

$$\mu''(\mathbf{x}) = prod\{\mu_2(\mathbf{x}), \mu_3(\mathbf{x})\} = \begin{cases} 0 & \mathbf{x} < 3 \\ 0.049(\mathbf{x}^2 - 4\mathbf{x} + 3) & 3 \le \mathbf{x} < 4 \\ 0.09(\mathbf{x} - 4)^2 + 0.216(\mathbf{x} - 4) + 0.14 & 4 \le \mathbf{x} < 6 \\ 0.003\mathbf{x}^2 - 0.23\mathbf{x} + 0.116 & 6 \le \mathbf{x} < 7 \\ 0.01\mathbf{x}^2 + 0.03\mathbf{x} + 0.02 & 7 \le \mathbf{x} < 8 \\ 0.1(\mathbf{x} - 7) + 0.8 & 8 \le \mathbf{x} < 9 \\ 1 & \mathbf{x} \ge 9 \end{cases}$$

Therefore, these membership functions are not suitable in our calculations. In the previous section we introduce a single formula membership function that help us to use product t-norm. In FLP problems, usually a membership function is dependent to a linear combination of n independent variables and we should use a simple changing variable to obtain a single variable membership function. For more statement, let

$$\mu(\mathbf{x}) = \begin{cases} 1 & B_i\mathbf{x} \le d_i \\ 1 - \frac{B_i\mathbf{x} - d_i}{p_i} & d_i < B_i\mathbf{x} \le d_i + p_i \\ 0 & B_i\mathbf{x} > d_i + p_i \end{cases}$$

By choosing $z_i = B_i\mathbf{x}$, we have

$$\mu(z_i) = \mu(B_i\mathbf{x}) = \begin{cases} 1 & z_i \le d_i \\ 1 - \frac{z_i - d_i}{p_i} & d_i < z_i \le d_i + p_i \\ 0 & z_i > d_i + p_i \end{cases}$$

Thus, according to previous section, smooth membership function is as follows:

$$\mu(z_i) = \frac{1}{\pi}\arctan\left(p(z_i)\right) + \frac{1}{2} \quad , \quad i = 1, 2, \ldots, m+1$$

Now, we can use max-product operator to obtain a crisp solution of FLP. On the other word, we maximize product of membership functions. That is,

$$\max \left\{ \mu(z) = \prod_{i=1}^{m+1} \mu(z_i) \right\} \tag{4.1}$$

Note that in this approach we don't have any constraint.

5. NUMERICAL EXAMPLES

In the section, we solving some numerical examples by modified $\max - \min$ and Max-Product approaches.

Example 5.1. *Consider the following Lp*

$$\begin{aligned} &\max\ 3x_1 + x_2 \\ &\ s.t. \\ &\qquad x_1 + 2x_2 \leq 16 \\ &\qquad 2x_1 + x_2 \leq 17 \\ &\qquad x_1, x_2 \geq 0 \end{aligned}$$

The solution is $x_1 = 8.5, x_2 = 0, z^* = 25.5$. Now, we considered the lower bounds and spread of the tolerance interval:

$$\begin{array}{lll} d_1 = 30 & d_2 = 16 & d_3 = 17 \\ p_1 = 6 & p_2 = 3 & p_3 = 4 \end{array}$$

Corresponding membership functions are as follows:

$$\mu_1(\mathbf{x}) = \begin{cases} 1 & x_1 + 2x_2 \leq 16 \\ 1 - \frac{x_1+2x_2-16}{3} & 16 < x_1 + 2x_2 \leq 19 \\ 0 & x_1 + 2x_2 \geq 19 \end{cases}$$

$$\mu_2(\mathbf{x}) = \begin{cases} 1 & 2x_1 + x_2 \leq 17 \\ 1 - \frac{2x_1+x_2-17}{4} & 17 < 2x_1 + x_2 \leq 21 \\ 0 & 2x_1 + x_2 \geq 21 \end{cases}$$

$$\mu_3(\mathbf{x}) = \begin{cases} 1 & 3x_1 + x_2 \geq 30 \\ 1 + \frac{(3x_1+x_2)-30}{6} & 24 < 3x_1 + x_2 \leq 30 \\ 0 & 3x_1 + x_2 \leq 24 \end{cases}$$

By $\max - \min$ approach, we have

$$\begin{aligned} &\max\ \lambda \\ &\ s.t. \\ &\qquad 3x_1 + x_2 - 6\lambda \geq 24 \\ &\qquad x_1 + 2x_2 + 3\lambda \leq 19 \\ &\qquad 2x_1 + x_2 + 4\lambda \leq 21 \\ &\qquad x_1 + 2x_2 \geq 16 \\ &\qquad 2x_1 + x_2 \geq 17 \\ &\qquad 3x_1 + x_2 \leq 30 \\ &\qquad x_1, x_2, \lambda \geq 0 \end{aligned}$$

The solution is $x_1 = 7.474, x_2 = 4.263, z^* = 26.685$ and degree of aspiration is $\lambda^* = 0.447$.

For solving this problem by using Max-Product approach, we set

$$z_1 = x_1 + 2x_2, \qquad z_2 = 2x_1 + x_2, \qquad z_3 = 3x_1 + x_2$$

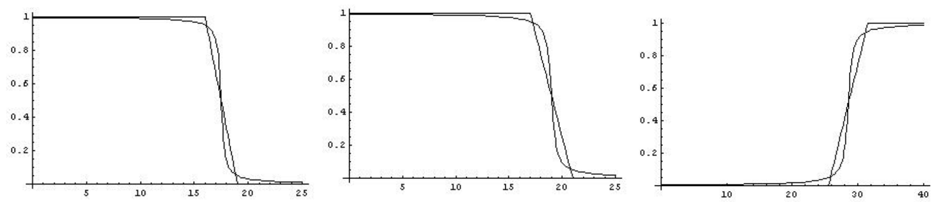

FIGURE 5. membership functions $\mu(z_1), \mu(z_2), \mu(z_3)$

Then

$$\mu_1(z_1) = \begin{cases} 1 & z_1 \leq 16 \\ 1 - \frac{z_1 - 16}{3} & 16 < z_1 \leq 19 \\ 0 & z_1 \geq 19 \end{cases}$$

Consider main points as $S = (16, 6.314), (19, -6.314)$, then
$p(z_1) = -4.2093z_1 + 73.6628$.

$$\mu_2(z_2) = \begin{cases} 1 & z_2 \leq 17 \\ 1 - \frac{z_2 - 17}{4} & 17 < z_2 \leq 21 \\ 0 & z_2 \geq 21 \end{cases}$$

Consider main points as $S = (17, 6.314), (21, -6.314)$, then
$p(z_2) = -3.157z_2 + 59.983$.

$$\mu_3(z_3) = \begin{cases} 1 & z_3 \geq 30 \\ 1 + \frac{z_3 - 30}{6} & 24 < z_3 \leq 30 \\ 0 & z_3 \leq 24 \end{cases}$$

Consider main points as $S = (30, 6.314), (24, -6.314)$, then
$p(z_3) = 2.1047z_3 - 56.826$.
Now smooth membership functions are as follows:

$$\begin{aligned} \mu_1(z_1) &= \frac{1}{\pi} \arctan(-4.2093z_1 + 73.6628) + \frac{1}{2} \\ \mu_2(z_2) &= \frac{1}{\pi} \arctan(-3.157z_2 + 59.983) + \frac{1}{2} \\ \mu_3(z_3) &= \frac{1}{\pi} \arctan(2.1047z_3 - 56.826) + \frac{1}{2} \end{aligned}$$

Figure (5) shows these membership functions. By solving non-linear programming (4.1), optimal solution is as follows:

$$x_1 = 9.2498 \quad , \quad x_2 = 0 \quad , \quad z^* = 27.7494$$

And degree of aspiration is 0.625.

Example 5.2. *Consider the example (2.1), here, we solve it by Max-Product approach. Smooth membership functions are as follows:*

$$\mu(z_1) = \frac{1}{\pi}\arctan(-2.5256z_1 + 99.7612) + \frac{1}{2}$$
$$\mu(z_2) = \frac{1}{\pi}\arctan(1.2628z_2 - 208.362) + \frac{1}{2}$$
$$\mu(z_3) = \frac{1}{\pi}\arctan(0.12628z_3 - 157.85) + \frac{1}{2}$$
$$\mu(z_4) = \frac{1}{\pi}\arctan(2.1047z_4 - 6.314) + \frac{1}{2}$$

The optimal solution is as follows:

$$x_1 = 10.72754, \quad x_2 = x_3 = 0, \quad x_4 = 69.31763, \quad z^* = 3847615$$

6. CONCLUSIONS

In this paper, we present an approach to making single formula for all types of membership functions with using function arctan and polynomials. These functions are many useful and may be used in all types of fuzzy programming. Here, we use them for fuzzy linear programming with fuzzy constraints. Since we solved these problems by using the max $-min$ operator and $Max - Product$ operator, so when we use the min operator and the algebraic product as a t-norm. Obtained results show that this methods are not efficiency. So we use Max-Product operator and present a new approach for this group of fuzzy linear programming problems, that present the optimal solution of them.

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